G-MGF based performance analysis of coherent FSO systems with multiple receivers

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This letter investigates the performance of coherent free-space optical systems with multiple receivers especially through the generalised moment-generating function based approach. Specifically, we first derive a closed-form expression of the generalised moment-generating function of the signal-to-noise ratio for the coherent free-space optical systems. The derived generalised moment-generating function formula is further utilised to effectively analyse the several performance metrics such as amount of fading and ergodic capacity. The accuracy of our theoretical analyses are corroborated via some numerical results.

Introduction: The coherent free-space optical (FSO) communication systems have gained the enormous attention due to the benefits of the easy deployment, huge bandwidth and license-free optical medium, which provides the significant performance improvements compared to the intensity modulation/direct detection [1, 2]. Accordingly, many experts have actively investigated the practical applications and theoretical characteristics of FSO systems. However, the atmospheric turbulence-induced fading or scintillation results in the major performance impairments of FSO systems. To this end, the several studies for the coherent FSO systems with multiple receive apertures have been introduced to utilise the spatial diversity techniques [1–3]. The authors of [3] investigated the diversity gain and diversity-multiplexing trade-off of coherent FSO systems with multiple receive apertures for the channel model that reflects the combined effects of turbulence-induced phase distortion and amplitude fluctuation introduced in [4]. The average error probability and ergodic capacity (EC) of the coherent FSO systems along with system and channel models were analysed in [3, 4].

It has been well acknowledged that the generalised moment-generating function (G-MGF) can be adopted to effectively calculate the achievable performance metrics of interest [5–8]. Thus, in this letter, we analyse the performance of coherent FSO systems with multiple receivers based on the G-MGF, where the channel model and system model [1–4] are adopted. Specifically, contrary to the previous works given in [1–4], we first derive a closed-form expression for the G-MGF of the signal-to-noise ratio (SNR) for the coherent FSO systems, from which the moments of the SNR can be readily attained. We then investigate the applicability of the G-MGF in two different scenarios, such as (1) amount of fading (AoF) and (2) EC, in the latter of which the compact asymptotic behaviour in the high-SNR regimes is further evaluated.

System model: We consider the FSO system with M heterodyne receivers introduced in [1–3], where the phase distortion caused by the atmospheric turbulence at the receivers is assumed to be compensated by the modal compensation with Zernike polynomials. Then, the output of signal at each receiver is $y(t)=\sum_{k=1}^{M}x_k(t)+\sum_{k=1}^{M}n_k(t)$, where $x_k(t)$ and $n_k(t)$ represent the data part carrying information and the noise part, respectively, the latter of which is assumed to be modelled as additive white Gaussian noise [1–3]. Accordingly, the SNR of the output of the signal at receiver is given by

$$\gamma=\frac{\gamma_0}{M}\sum_{k=1}^{M}\alpha_k^2, \tag{1}$$

where γ_0 is the SNR in the absence of the turbulence. Additionally, $\alpha_k=\alpha_{R,k}+j\alpha_{I,k}$ is the effective fading coefficient modelling the channel of kth diversity branch with satisfying $\alpha_k=|\alpha_k|$, where $\alpha_{R,k}$ and $\alpha_{I,k}$ are supposed to follow Gaussian distribution, that is $\alpha_{R,k}\sim\mathcal{N}(\overline{\alpha_R},\sigma_R^2)$ and $\alpha_{I,k}\sim\mathcal{N}(\overline{\alpha_I},\sigma_I^2)$. The fading parameters $\overline{\alpha_R}$, $\overline{\alpha_I}$, σ_R^2 and σ_I^2 are defined as follows [1–3]:

$$\begin{aligned}\overline{\alpha_R}&=\mathrm{e}^{-\frac{\sigma_\chi^2+\sigma_\phi^2}{2}},\quad \overline{\alpha_I}=0\\ \sigma_R^2&=\frac{1}{2N}\left(1+\mathrm{e}^{-2\sigma_\phi^2}-2\mathrm{e}^{-\sigma_\chi^2-\sigma_\phi^2}\right)\\ \sigma_I^2&=\frac{1}{2N}\left(1-\mathrm{e}^{-2\sigma_\phi^2}\right),\end{aligned} \tag{2}$$

where $\sigma_\chi^2=0.307k^{7/6}Z^{11/6}C_n^2$ is the log-amplitude variance, $\sigma_\phi^2=C_J(D_k/r_0)$ is the residual phase variance after the compensation of J Zernike terms over each receive aperture with diameter $D_k=D/\sqrt{M}$, D denotes the receive aperture diameter of the benchmark single receiver system, C_J is the Zernike–Jolmogoroff residual error, $r_0=1.68(C_n^2Zk^2)^{-3/5}$ is the Fried parameter corresponding to the wavefront coherence diameter with C_n^2 being the refractive index structure constant, Z being the propagation distance and k being the wavenumber, and $N=\{1.09(\rho_0/D_k)^2\Gamma(6/5,1.08(D_k/\rho_0)^{5/3})\}^{-1}$ is the number of statistically independent patches or cells with $\Gamma(\cdot)$ being the lower incomplete gamma function [9] and $\rho_0\approx(3.04/C_J)^{3/5}0.286J^{-0.362}r_0$ being the generalised Fried parameter after the compensation of J Zernike terms. Given that N is large enough, α_k is known to follow Rician distribution with probability density function (PDF) [1–3]

$$f_{\alpha_k}(\alpha_k)=\frac{2\alpha_k(1+K)}{\overline{\alpha^2}}\mathrm{e}^{-K}\mathrm{e}^{-\frac{(1+K)\alpha_k^2}{\overline{\alpha^2}}}I_0\left(2\alpha_k\sqrt{\frac{(1+K)K}{\overline{\alpha^2}}}\right), \tag{3}$$

where $\overline{\alpha^2}=\sigma_R^2+\sigma_I^2+\overline{\alpha_R}^2$, $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν [9], and the parameter K is the ratio of the strength of the coherent component to that of the incoherent one with satisfying the condition $K^{-1}=\overline{\alpha^2}(\overline{\alpha_R}^4+2\overline{\alpha_R}^2(\sigma_I^2-\sigma_R^2)-(\sigma_I^2-\sigma_R^2)^2)^{-1/2}-1$.

Then, by the definition of the SNR γ given in (1) and employing a simple variable change, the PDF of the SNR can be derived as [1–3]

$$\begin{aligned}f_\gamma(\gamma)=&\frac{(1+K)M}{\overline{\alpha^2}\gamma_0}\left(\frac{(1+K)\gamma}{K\overline{\alpha^2}\gamma_0}\right)^{\frac{(M-1)}{2}}\\ &\times\mathrm{e}^{-KM-\frac{(1+K)M\gamma}{\overline{\alpha^2}\gamma_0}}I_{M-1}\left(2M\sqrt{\frac{K(1+K)\gamma}{\overline{\alpha^2}\gamma_0}}\right).\end{aligned} \tag{4}$$

G-MGF based analysis: The G-MGF of a random variable X is defined as [5–8]

$$\mathcal{M}_X^{(n)}(s)\triangleq\mathbb{E}\{X^n\mathrm{e}^{Xs}\}=\int_0^\infty x^n\mathrm{e}^{xs}f_X(x)dx, \tag{5}$$

where $f_X(x)$ is the PDF of X and $\mathbb{E}\{\cdot\}$ denotes the expectation operator. Then, if $n\in\mathbb{Z}^+$, the G-MGF equals the nth order derivative of the MGF $\mathcal{M}_X(s)\triangleq\mathbb{E}\{\mathrm{e}^{Xs}\}=\mathcal{M}_X^{(0)}$, and the nth order moment of X can be obtained as $\mathbb{E}\{X^n\}=\mathcal{M}_X^{(n)}(0)$.

Thus, from the definition of the G-MGF given in (5), the G-MGF of the coherent FSO systems with M multiple receivers can be formulated as

$$\begin{aligned}\mathcal{M}_\gamma^{(n)}(s)&=\mathbb{E}\{\gamma^n\mathrm{e}^{\gamma s}\}\\ &=\int_0^\infty\gamma^n\mathrm{e}^{\gamma s}f_\gamma(\gamma)d\gamma\\ &=\int_0^\infty\mu\left(\frac{\mu}{KM}\right)^{\frac{M-1}{2}}\mathrm{e}^{-KM}\gamma^{\frac{M-1}{2}+n}\mathrm{e}^{(s-\mu)\gamma}\\ &\quad\times I_{M-1}(2\sqrt{KM\mu}\sqrt{\gamma})d\gamma\\ &\overset{(a)}{=}\mu\left(\frac{\mu}{KM}\right)^{\frac{M-1}{2}}\mathrm{e}^{-KM}\end{aligned}$$

$$\times \int_0^{\infty} 2x^{M+2n} e^{(s-\mu)x^2} I_{M-1}(2\sqrt{KM\mu}x)dx$$

$$\overset{(b)}{=} \frac{\Gamma(M+n)}{\Gamma(M)} \frac{\mu^M e^{-KM}}{(\mu - s)^{M+n}} {}_1F_1\left(M+n, M, \frac{KM\mu}{\mu - s}\right), \quad (6)$$

where $\mu = (1+K)M/(\overline{\alpha^2}\gamma_0)$, $\Gamma(x)$ represents the Gamma function [9], ${}_1F_1(x, y, z)$ denotes Kummer confluent hypergeometric function, (a) can be obtained by defining $x = \sqrt{\gamma}$, and (b) can be derived by using the formula [10, equations (9.4-11)]. Moreover, since the number M of multiple receivers and the parameter n for the G-MGF in (5) are positive and non-negative integers, respectively, that is, $M \in \mathbb{Z}^+$ and $n \in \{0\} \cup \mathbb{Z}^+$, the G-MGF for the coherent FSO systems with multiple receivers in (6) can be rewritten as

$$\mathcal{M}_{\gamma}^{(n)}(s) = \frac{\Gamma(M+n)}{\Gamma(M)} \frac{1}{\mu^n} \left(\frac{1}{1 - \frac{s}{\mu}}\right)^{M+n}$$

$$\times e^{-KM\left(1 - \frac{1}{1-\frac{s}{\mu}}\right)} \sum_{p=0}^{n} \frac{(-n)_p \left(-\frac{KM}{1-\frac{s}{\mu}}\right)^p}{p!(M)_p}, \quad (7)$$

where $(x)_p \triangleq \Gamma(x+p)/\Gamma(x)$ denotes the Pochhammer symbol, and for $x \in \mathbb{Z}^+ \wedge y \in \mathbb{Z}^+ \wedge y \leq x$, ${}_1F_1(x, y, z)$ can be expressed as [11, equation (07.20.03.0025.01)]

$${}_1F_1(x, y, z) = e^{-z} \sum_{p=0}^{x-y} \frac{(y-x)_p(-z)^p}{p!(y)_p}. \quad (8)$$

Therefore, the nth moment of the SNR γ of coherent FSO systems can be also obtained from (7) as

$$\mathbb{E}\{\gamma^n\} = \left(\mathcal{M}_{\gamma}^{(n)}(s)\right|_{s=0}$$

$$= \frac{\Gamma(M+n)}{\Gamma(M)} \frac{e^{-KM}}{\mu^n} {}_1F_1(M+n, M, KM)$$

$$= \frac{\Gamma(M+n)}{\Gamma(M)} \frac{e^{-KM}}{\mu^n} \sum_{p=0}^{n} \frac{(-n)_p(-KM)^p}{p!(M)_p}. \quad (9)$$

It is worthy to note that equations (6), (7) and (9) are new in the literature to the best of our knowledge.

Applications to performance analysis: We here illustrate the applicability of the G-MGF for the SNR γ of coherent FSO systems in the different scenarios: (1) AoF and (2) EC. Additionally, it is noticeable that the G-MGF can be further applicable to the various performance evaluations, for example, error probability analysis, outage probability with interference, energy detection, and physical layer security [5–8].

A. Amount of fading: The AoF is generally used as fundamental performance metric to describe the fading severity of channel model, which aims to define the distribution of the SNR of the received signal, and thus can be defined as [12]

$$\text{AoF}_{\gamma} \triangleq \frac{\text{Var}(\gamma)}{(\mathbb{E}\{\gamma\})^2} = \frac{\mathbb{E}\{\gamma^2\} - (\mathbb{E}\{\gamma\})^2}{(\mathbb{E}\{\gamma\})^2} = \frac{\mathbb{E}\{\gamma^2\}}{(\mathbb{E}\{\gamma\})^2} - 1, \quad (10)$$

where $\text{Var}(\gamma)$ denotes the variance of the SNR γ. Using (6) and (9), we can have

$$\mathbb{E}\{\gamma\} = \left(\mathcal{M}_{\gamma}^{(1)}(s)\right|_{s=0}$$

$$= \frac{\Gamma(M+1)}{\Gamma(M)} \frac{e^{-KM}}{\mu} {}_1F_1(M+1, M, KM)$$

$$= \frac{(K+1)M}{\mu}, \quad (11a)$$

$$\mathbb{E}\{\gamma^2\} = \left(\mathcal{M}_{\gamma}^{(2)}(s)\right|_{s=0}$$

$$= \frac{\Gamma(M+2)}{\Gamma(M)} \frac{e^{-KM}}{\mu^2} {}_1F_1(M+2, M, KM)$$

$$= \frac{M}{\mu^2}\left[(K+1)^2 M + 2K + 1\right]. \quad (11b)$$

Hence, by substituting (11a) and (11b) into (10), the AoF for the SNR γ of coherent FSO systems can be readily obtained as

$$\text{AoF}_{\gamma} = \frac{1}{(K+1)^2 M}\left[(K+1)^2 M + 2K + 1\right] - 1$$

$$= \frac{2K+1}{(K+1)^2 M}, \quad (12)$$

where it is apparent that the AoF depends only on the ratio K of strength of the coherent component to that of the incoherent one and the number M of heterodyne receivers of FSO systems.

B. Ergodic capacity: The EC represents the quantity of information transferred through the time-variant channels. Based on the derived G-MGF in (6), the EC [bps/Hz] for the coherent FSO systems with multiple receivers can be expressed as the expected value of $\log_2(1+\gamma)$, that is, [13, 14]

$$\mathcal{C} = \mathbb{E}\{\log_2(1+\gamma)\}$$

$$= \int_0^{\infty} \log_2(1+\gamma) f_{\gamma}(\gamma) d\gamma$$

$$\overset{(a)}{=} -\frac{1}{\ln 2} \times \int_0^{\infty} E_i(-s)\mathcal{M}_{\gamma}^{(1)}(-s)ds$$

$$\approx \frac{1}{\ln 2} \sum_{n=0}^{N} v_n E_i(-s_n)\left(\left[\mathcal{M}_{\gamma}^{(1)}(-s)\right]\right|_{-s \to s_n}, \quad (13)$$

where N is any positive integer, $E_i(\cdot)$ is exponential integral function [9], (a) can be obtained with the G-MGF in (6) by following [13, equation (7)], and the coefficients v_n and s_n are given as [14]

$$v_n = \frac{\pi^2 \sin\left(\frac{2n-1}{2N}\pi\right)}{4N \cos^2\left(\frac{\pi}{4}\cos\left(\frac{2n-1}{2N}\pi\right) + \frac{\pi}{4}\right)}, \quad (14)$$

$$s_n = \tan\left(\frac{\pi}{4}\cos\left(\frac{2n-1}{2N}\pi\right) + \frac{\pi}{4}\right). \quad (15)$$

Furthermore, for the high-SNR regime, the compact asymptotic EC can also be formulated by utilising the nth moment of SNR γ as [15, equation (22)]

$$\mathcal{C}^{\infty} = \frac{1}{\ln 2}\left(\frac{\partial}{\partial n}\mathbb{E}\{\gamma^n\}\right|_{n=0}$$

$$= \frac{1}{\ln 2}\left(\frac{\partial}{\partial n}\mathcal{M}_{\gamma}^{(n)}(0)\right|_{n=0}$$

$$\overset{(a)}{=} \frac{1}{\ln 2}\left(e^{-KM} {}_1F_1^{(1,0,0)}(M, M, KM) - \log\mu + \psi^{(0)}(M)\right)$$

$$= \frac{1}{\ln 2}\left(e^{-KM}\sum_{p=0}^{\infty} \frac{(M)_p \psi(M+p)(KM)^p}{p!(M)_p} - e^{-KM}\psi(M)\right.$$

$$\left.\times {}_1F_1(M, M, KM) - \log\left(\frac{(1+K)M}{\overline{\alpha^2}\gamma_0}\right) + \psi^{(0)}(M)\right)$$

$$= \frac{1}{\ln 2}\left(\sum_{p=0}^{\infty} \frac{e^{-KM}\psi(M+p)(KM)^p}{p!} - \log\left(\frac{(1+K)M}{\overline{\alpha^2}\gamma_0}\right)\right), \quad (16)$$

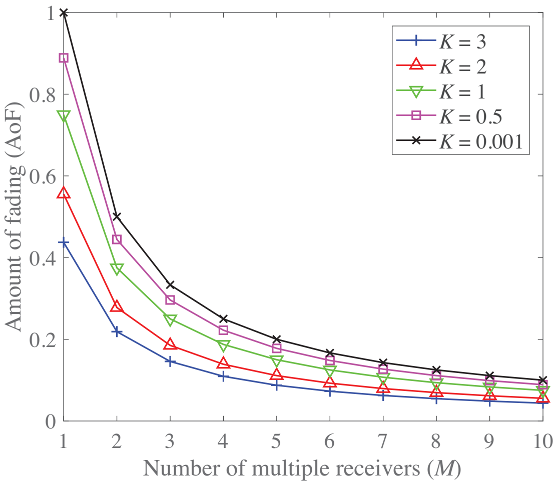

Fig. 1 *AoF as a function of number of multiple receivers M with the different ratio K*

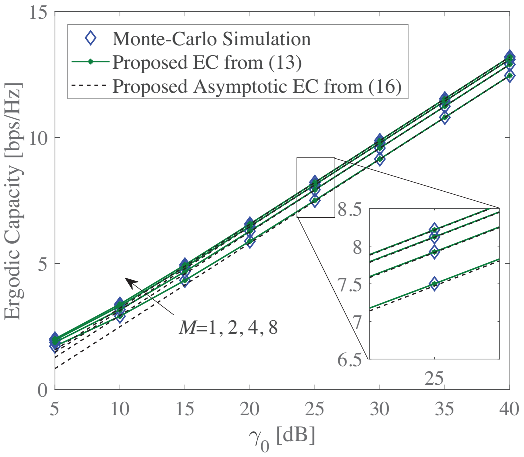

Fig. 2 *EC as a function of SNR γ_0 with the different number of multiple receivers M*

where $\psi(\cdot)$ is the Euler's digamma function [9], $\psi^{(0)}(\cdot)$ is the 0th derivative of digamma function to satisfy $\psi^{(0)}(x) = \psi(x)$, ${}_1F_1^{(1,0,0)}(a, b, x)$ denotes the derivative of ${}_1F_1(a, b, x)$ with respect to a, and (a) is attained by using [11, equation (07.20.20.0001.01)]. We note that although the EC achieved by the considered systems was theoretically investigated in [1] and the simpler EC expression was given in [2], we attempt here to derive the novel concise EC formula especially by exploiting the G-MGF in (6) and thus to demonstrate the usefulness of the G-MGF newly derived in this letter. Moreover, we can also find that the asymptotic EC expression in (16) is equivalent to that given in [2, equation (9)], however, where the PDF-based approach was used.

Numerical results: In this section, we present some numerical results for the AoF and EC of the coherent FSO systems with multiple receivers eventually to verify the usability of the G-MGF derived in this letter. For simplicity, we consider the coherent FSO system with $\overline{\alpha^2} = 1$. Additionally, in order to calculate the infinite summation in (16), we restrict to $p = 0, \ldots, 50$.

In Figure 1, we analyse the AoF as a function of number of multiple receivers M with the different ratio K (i.e. $K = 0.001, 0.5, 1, 2, 3$) for the coherent FSO system. As shown in the figure, the AoF decreases as both the number of multiple receivers M and the ratio K increase. It is also observed that the AoF is significantly affected by the number of receive apertures especially for the small M.

Figure 2 illustrates the EC obtained from (13) and its corresponding asymptotic expression in (16) as a function of SNR γ_0 with the ratio $K = 0.001$ for the different number of multiple receivers M (i.e. $M = 1, 2, 4, 8$). In the figure, we can observe that the numerical results from the proposed formula (i.e. (13)) exactly coincide, in all SNR regions, with those from the Monte-Carlo simulation, and the results from the derived asymptotic expression (i.e. (16)) also tightly converge to the corresponding ones from (13) in the high-SNR regime. This evidently implies that by using the derived formulas, the EC performance of the coherent FSO systems can be easily but precisely predicted for different system parameters and channel conditions.

Conclusions: We have comprehensively evaluated the performance achieved by the coherent FSO systems with multiple receivers through judiciously exploiting the G-MGF. To this end, we have first derived the novel closed-form expression for the G-MGF of the SNR, and then explored its usefulness and applicability for two different performance metrics, that is, AoF and EC, whose achievable performances were explicitly verified through various numerical results.

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