Research Article



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Advanced non-dimensional dynamic influence function method considering the singularity of the system matrix for accurate eigenvalue analysis of membranes

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Abstract

An advanced non-dimensional dynamic influence function method (NDIF method) for highly accurate free vibration analysis of membranes with arbitrary shapes is proposed in this paper. The existing NDIF method has the weakness of not offering eigenvalues and eigenmodes in the low frequency range when the number of boundary nodes of an analyzed membrane is increased to obtain more accurate result. This paper reveals that the system matrix of the membrane becomes singular in the lower frequency range when the number of the nodes increases excessively. Based on this fact, it provides an efficient way to successfully overcome the weaknesses of the existing NDIF method and still maintain its accuracy. Finally, verification examples show the validity and accuracy of the advanced NDIF method proposed.

Keywords

Non-dimensional dynamic influence function method, eigenvalue, membrane vibration, mode shape, natural frequency, free vibration, matrix singularity

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Introduction

The non-dimensional dynamic influence function method (NDIF method) are for the first time introduced by the author for the extraction of highly accurate eigenvalues and mode shapes of membranes arbitrary shapes.¹ Furthermore, the NDIF method was extended to arbitrarily shaped acoustic cavities with the rigid-wall boundary² and plates with various boundary conditions.^{3–5} Until recently, in-depth studies have been conducted by the author to overcome the frequencydependent problem of the system matrix in the NDIF method.^{6–11}

Although a vast literature exists on analytical and semi-analytical methods for obtaining accurate eigenvalues of membranes having no exact solution and the author has scrutinized the vast literature, only relatively recent studies are introduced in the paper. Gol'dshtein and Ukhov¹² obtained estimates for the first non-trivial eigenvalues of membranes in conformal regular domains. Zheng et al.¹³ solved the nonlinear free vibration problem of axisymmetric circular membrane by both the Galerkin method and the large deflection theory. Ouakad¹⁴ extracted the natural frequencies and mode shapes of rectangular membranes with rounded

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edges utilized in practical engineering applications. Bahrami and Teimourian¹⁵ proposed a systematic method for free vibration analysis of non-uniform annular and circular membranes and obtained the natural frequencies of the membranes using the wave propagation approach. Siedlecka et al. obtained eigenfrequencies of composite circular and annular membranes with nonuniform material property.16 Yakhno and Ozdek¹⁷ solved the free vibration problem of a composite circular membrane whose density and tension change piecewisely by approximately commutating the Green's function. Lastly, Wang and Chien¹⁸ Ming derived exact solutions for eigenvalue analysis of circular, annular and sector membranes and investigated the effect of shape variables of the membranes on natural frequencies. As a result of reviewing the vast literature including the recent studies as mentioned above, it is confirmed that there are no studies that provide more accurate eigenvalues of arbitrarily shaped membranes than the NDIF method.

On the other hand, the NDIF method has the advantage of needing a small amount of numerical calculation and offering much more accurate results than other numerical analysis methods such as the finite element method¹⁹ and the boundary element method.²⁰ This is because the basis functions of the NDIF method exactly satisfy the governing differential equation and at the same time the boundary of an analyzed membrane is discretized by a small number of nodes unlike the two methods.^{19,20}

However, the NDIF method has the disadvantage that lower-order eigenvalues and eigenmodes are not extracted if more boundary nodes are used to obtain more accurate higher-order ones. For this reason, it also has the disadvantage that the membrane should be analyzed once more after decreasing the number of the boundary nodes if lower-order eigenvalues are not extracted. In the paper, the reason why low-order eigenvalues and eigenmodes are not obtained is first identified and then an advanced NDIF method is proposed to achieve both low-order and higher-order eigenvalues (including eigenmodes) without losing the accuracy of the existing NDIF method when using many nodes.

Existing NDIF method reviewed

In Figure 1, when a unit displacement is excited at one point P_k in an infinite membrane, displacement at another point P is given by the non-dimensional dynamic influence function (NDIF) as follows:

$$NDIF = J_0(\Lambda |\mathbf{r} - \mathbf{r}_k|) \tag{1}$$

which is the Bessel function of the first kind of order zero.^{1,8} In equation (1), $|\mathbf{r} - \mathbf{r}_k|$ means the distance between *P* and *P_k* of which the position vectors are \mathbf{r}



Figure 1. Infinite membrane with harmonic excitation points $(P_1, P_2, ..., P_N)$ distributed along the fictitious contour (dotted line) that is the same shape as the finite-sized membrane of interest.

and \mathbf{r}_k , respectively. Also, $\Lambda = \omega/\sqrt{T/\rho}$ is a frequency parameter where ω , *T*, and ρ are the angular frequency, the uniform tension per unit length, and the mass per unit area, respectively. The NDIF satisfies Helmholtz equation (equation (2)), which is the governing equation for free vibration of the finite-sized membrane depicted by the dotted line in Figure 1.

$$\nabla^2 W(\mathbf{r}) + \Lambda^2 W(\mathbf{r}) = 0, \qquad (2)$$

where $W(\mathbf{r})$ is the transverse displacement of the finitesized membrane.

For free vibration analysis of the finite-sized membrane shown in Figure 1, N boundary points are first distributed at points P_1 , P_2 , ..., P_N along the fictitious boundary (dotted line) of the membrane. If the amplitudes of displacements excited simultaneously at points P_1 , P_2 , ..., P_N are A_1 , A_2 , ..., A_N , respectively, the displacement response at the point P inside the membrane can be obtained by the linear combination of NDIFs as follows.

$$W(\mathbf{r}) = \sum_{k=1}^{N} A_k J_0(\Lambda |\mathbf{r} - \mathbf{r}_k|), \qquad (3)$$

In the NDIF method, equation (3) is utilized as an approximate solution for free vibration of the finitesized membrane, which has an arbitrary shape. It should be noticed that equation (3) that is the linear combination of NDIFs naturally satisfies the governing equation.

Next, if the boundary of the membrane is fixed, a discrete boundary condition for points P_1, P_2, \ldots, P_N is given by

	Existing NDIF method		Exact solution ²²	FEM (ANSYS) ⁶			Advanced NDIF
	N = 16	N = 32		1089 nodes	289 nodes	49 nodes	N = 32
Λ_1	4.3633 (0.00)	None	4.3633	4.3651 (0.04)	4.3703 (0.16)	4.4133 (1.15)	4.3633 (0.00)
Λ_2	6.2927 ([–] 0.003)	None	6.2929	6.3006 (0.12)	6.3240 (0.49)	6.5166 (3.55)	6.2929 (0.00)
Λ_3	7.4549 (–0.01)	None	7.4560	7.4669 (0.15)	7.4996 (0.58)	7.7682 (4.19)	7.4560 (0.00)
Λ_4	8.6001 (0.06)	8.5947 (0.00)	8.5947	8.6213 (0.31)	8.7013 (1.24)	9.1287 (6.21)	8.5947 (0.00)
Λ_5	8.7101 (–0.Í9)	8.7266 (0.00)	8.7266	8.7407 (0.16)	8.7828 (0.64)	9.3523 (7.17)	8.7266 (0.00)
Λ_6	None	10.5083 (0.00)	10.5083	10.5370 (0.27)	10.6234 (1.10)	11.3284 (7.80)	10.5083 (0.00)
Λ_7	10.7881 (-0.06)	10.7943 (0.00)	10.7943	10.8313 (0.34)	10.9428 (1.38)	11.8467 (9.75)	10.7943 (0.00)
Λ_8	None	I I .0384 (0.00)	11.0384	II.1029 (0.58)	II.2974 (2.35)	12.7802 (15.78)	II.0384 (0.00)

Table I. Eigenvalues of the rectangular membrane obtained by the existing NDIF method, the exact solution,²² FEM (ANSYS),⁶ and the advanced NDIF method (parenthesized values: errors (%) relative to the exact solution).

$$W(\mathbf{r}_i) = 0, i = 1, 2, \dots, N$$
 (4)

where \mathbf{r}_i denotes the position vector for point P_i . Then, substituting the approximate solution into the discrete boundary condition gives

$$W(\mathbf{r}_{i}) = \sum_{k=1}^{N} A_{k} J_{0}(\Lambda |\mathbf{r}_{i} - \mathbf{r}_{k}|) = 0, i = 1, 2, ..., N.$$
(5)

Finally, equation (5) may be written in a simple matrix form:

$$\mathbf{SM}(\Lambda) \mathbf{A} = \mathbf{0},\tag{6}$$

where **SM**(Λ) is called *system matrix* in the NDIF method and is a function of the frequency parameter Λ . In equation (6), the contribution vector **A** represents the contribution strength of the NDIFs defined at each boundary point. The elements of the system matrix of which the size is $N \times N$ are calculated by

$$SM_{ik} = J_0(\Lambda |\mathbf{r}_i - \mathbf{r}_k|). \tag{7}$$

The condition for equation (6) to have a non-trivial solution is that the determinant of the system matrix becomes zero as follows.

$$\det(\mathbf{SM}(\Lambda)) = 0. \tag{8}$$

The *j* th eigenvalue can be calculated from equation (8) and the *j* th eigenvector can be obtained by applying the singular value decomposition²¹ to the system matrix **SM**(Λ_j) where Λ_j represents the *j* th eigenvalue. As mentioned earlier, equation (8) doesn't give lower-order eigenvalues and eigenmodes if the number of the boundary points (= boundary nodes) is increased to obtain higher-order ones. In the following section, an advanced NDIF method is presented to obtain both lower-order and higher-order eigenvalues and eigenmodes, regardless of the number of boundary nodes used.

Advanced NDIF method

Eigenvalue extraction characteristics of the existing NDIF method

A rectangular membrane as shown in Figure 2 is considered to investigate how eigenvalues extraction characteristics of the existing NDIF method change with the number of the boundary nodes. The horizontal and vertical lengths of the membrane are 1.2 and 0.9 m, respectively. First, the boundary of the membrane is discretized with 16 nodes as shown in Figure 2(a). For N = 16, logarithmic values of det(SM(Λ)) are plotted using the dotted line as a function of the frequency parameter in Figure 3 where values of the frequency parameter corresponding to the troughs (Λ_1 - Λ_5 , Λ_7) denote the eigenvalues of the membrane. Note that The logarithm values of det(SM(Λ)) are calculated by a commercial software such as MATLAB or Mathematica. It is confirmed in Table 1 that these eigenvalues agree well with the exact solution²² within 0.19% error but the sixth and eighth eigenvalues are not extracted because the number of the nodes is not enough.

To extract the sixth and eighth eigenvalues, the number of the boundary nodes is increased to 32 as shown in Figure 2(b). For N = 32, logarithmic values of $det(SM(\Lambda))$ are plotted using the solid line in Figure 3 where it may be seen that higher-order eigenvalues $(\Lambda_4 - \Lambda_8)$ including the sixth and eighth eigenvalues are extracted but lower-order eigenvalues $(\Lambda_1^-\Lambda_3)$ are not found. This is confirmed to be because there are cases where the values of det($SM(\Lambda)$) are negative in the low frequency parameter range where $\Lambda_1^-\Lambda_3$ exist. Further consideration of the problem that the determinant of the system matrix becomes negative will continue in the next section. In Table 1, note that the eigenvalues (Λ_{4-} Λ_8) obtained by the existing NDIF method using only 32 nodes exactly coincide with the exact solution,²² while the eigenvalues obtained by FEM (ANSYS) using 1089 nodes have relatively large errors with respect to the exact solution.



Figure 2. Rectangular membranes discretized with 16 and 32 boundary nodes, respectively: (a) N = 16 and (b) N = 32.



Figure 3. Logarithm values of the determinant of the system matrix obtained by the existing NDIF method for the rectangular membrane (dotted line: 16 nodes, solid line: 32 nodes).

In the following sections, the reason why the lowerorder eigenvalues are not found when the number of boundary nodes increases is revealed. Furthermore, an advanced NDIF method for extracting both the lowerorder and higher-order eigenvalues is presented on the basis of the reason revealed.

Rank of the system matrix

As confirmed in the previous section, the existing NDIF method provides far more accurate results than FEM (ANSYS), despite the use of much fewer nodes. However, it has the weakness of failing to extract lower-order eigenvalues when the number of nodes increases. To establish the cause of this weakness, the



Figure 4. Rank ($R(\Lambda)$) of the system matrix for the rectangular membrane (dotted line: 16 nodes, solid line: 32 nodes).

rank of the system matrix is first investigated in the frequency parameter range of interest.

Values of the rank $R(\Lambda)$ of the system matrix for the rectangular membrane are plotted as a function of the frequency parameter in Figure 4 where the dotted and solid lines are for N = 16 and N = 32, respectively. It may be seen in Figure 4 that values of the rank of the system matrix for N = 16 remain constant across the entire frequency parameter range, with a value of 16. On the other hand, it should be noticed in Figure 4 that values of the rank of the system matrix, 32, in the range where the frequency parameter is less than about 8.3. This means that rows or columns in the system matrix are not independent of each other and the system matrix becomes singular in the low frequency parameter range when using 32 nodes. As a result, the

determinant of the system matrix theoretically becomes zero but, in numerical computations of the NDIF method, results in a meaningless value close to zero. The reason why the system matrix becomes singular is because NDIF's used in equation (3) are not independent of each other and the distances between two adjacent nodes become closer. Furthermore, it turns out that the reason why some logarithm values for N = 32in Figure 3 do not exist in the low frequency parameter range is that values of the determinant of the system matrix are calculated as *negative* values close to zero.

Calculating the determinant of the singular system matrix

As revealed in the previous section, the determinant of the system matrix is calculated as a meaningless value in a frequency parameter range where the matrix becomes singular because the rank of the matrix is less than the order of the matrix. In the section, a practical way is proposed to ensure that the determinant of the system matrix has a valid value when the matrix is singular in numerical computations of the NDIF method.

A newly proposed way is based on the fact that the determinant of a matrix is equal to the product of eigenvalues of the matrix.²³ From this fact, the determinant of the system matrix may be calculated by

$$\det (\mathbf{SM}(\Lambda)) = \prod_{i=1}^{N} \lambda_{i}(\Lambda)$$
$$= \lambda_{1}(\Lambda)\lambda_{2}(\Lambda) \dots \lambda_{R(\Lambda)}(\Lambda)\lambda_{R(\Lambda)+1}(\Lambda) \dots \lambda_{N}(\Lambda)$$
(9)

where N is the order of the system matrix corresponding to the number of nodes used and $\lambda_i(\Lambda)$, which is a function of Λ , denotes the *i* th eigenvalue that satisfies the following the eigenvalue problem.

$$\mathbf{SM}(\Lambda)\mathbf{v}_i = \lambda_i(\Lambda)\mathbf{v}_i, i = 1, 2, \dots, N$$
(10)

where \mathbf{v}_i is the *i* th eigenvector for $\lambda_i(\Lambda)$.

It is explained using equation (9) why the determinant of the system matrix is calculated as a meaningless value in the low frequency parameter range of $\Lambda < 8.3$ for N = 32 in Figure 3. Although N eigenvalues are numerically calculated from equation (10) in the range of $\Lambda < 8.3$, the higher-order eigenvalues $\lambda_{R(\Lambda) + 1}$, $\lambda_{R(\Lambda) + 2}$, ..., λ_N become meaningless values because the rank $R(\Lambda)$ is less than N. Thanks to this fact, equation (9) calculated by the product of the N eigenvalues including the higher-order eigenvalues also has a meaningless value. Based on this findings, the determinant of the system matrix is calculated by the product of only the valid eigenvalues $\lambda_1, \lambda_2, ..., \lambda_{R(\Lambda)}$ excluding the invalid eigenvalues $\lambda_{R(\Lambda) + 1}, \lambda_{R(\Lambda) + 2}, ..., \lambda_N$ for $R(\Lambda) < N$ as follows.

$$\det \left(\mathbf{SM}(\Lambda) \right) = \prod_{i=1}^{R(\Lambda)} \lambda_i(\Lambda) = \lambda_1(\Lambda) \lambda_2(\Lambda) \dots \lambda_{R(\Lambda)}(\Lambda)$$
(11)

where $R(\Lambda)$ is given by the solid line of Figure 4. In order to help understand equation (11), it is assumed that $R(\Lambda)$ changes as expressed in equation (12) in the frequency parameter range of interest.

$$R(\Lambda) = \begin{cases} R_1 & , & C_0 \leq \Lambda < C_1 \\ R_2 & , & C_1 \leq \Lambda < C_2 \\ \vdots & \vdots \\ R_N & , & C_{N-1} \leq \Lambda < C_N \end{cases}$$
(12)

where $C_{i-1} \leq \Lambda < C_i$ denotes the *i* th frequency parameter range of which the minimum and maximum values are C_{i-1} and C_i , respectively, and R_i is a constant rank in the range $C_{i-1} \leq \Lambda < C_i$. If applying equation (12) to equation (11), equation (11) is expressed in an easy-to-understand manner as follows.

 $\det(\mathbf{SM}(\Lambda)) =$

$$\begin{cases}
\prod_{i=1}^{R_1} \lambda_i(\Lambda) = \lambda_1(\Lambda)\lambda_2(\Lambda) \cdots \lambda_{R_1}(\Lambda) , \quad C_0 \leq \Lambda < C_1 \\
\prod_{i=1}^{R_2} \lambda_i(\Lambda) = \lambda_1(\Lambda)\lambda_2(\Lambda) \cdots \lambda_{R_2}(\Lambda) , \quad C_1 \leq \Lambda < C_2 \\
\vdots \\
\prod_{i=1}^{R_N} \lambda_i(\Lambda) = \lambda_1(\Lambda)\lambda_2(\Lambda) \cdots \lambda_{R_N}(\Lambda) , \quad C_{N-1} \leq \Lambda < C_N
\end{cases}$$
(13)

Figure 5 shows the determinant curve for the rectangular membrane (N = 32) obtained by using equation (11) or (13). It may be seen in Figure 5 that the eight



Figure 5. Logarithm values of the determinant of the system matrix with discontinuity obtained by the advanced NDIF method using equations (11) for the rectangular membrane (N = 32).

Figure 6. Amount of change in logarithm values of the determinant equation (11) of the system matrix caused by the change of the rank for the rectangular membrane (N = 32).

eigenvalues including $\Lambda_1 - \Lambda_3$ are successfully extracted although 32 many nodes are used. The eight eigenvalues are tabularized in the last column of the Table 1 and are found to exactly match the exact solution²² without any error. Note that the eigenvalues obtained by FEM (ANSYS) have some errors with respect to the exact solution even though FEM uses much more nodes than the advanced NDIF method. It may be said that the advanced NDIF method proposed in the paper successfully offers accurate eigenvalues in the entire frequency parameter range irrespective of the number of nodes.

On the other hand, it is observed in Figure 5 that the determinant curve changes discontinuously. This is because the number of the eigenvalues $(\lambda_1, \lambda_2, ...,$ $\lambda_{R(\Lambda)}$) multiplied by each other in equation (11) or (13) changes due to the change of the rank $(R(\Lambda))$ as shown in Figure 4 (N = 32). Although the proposed method successfully gives accurate the lower-order eigenvalues as well as the higher-order ones as confirmed in Figure 5 and Table 1, a more complete method is proposed to eliminate the discontinuity of the determinant curve. First, the amount $\Delta \log (\det(\mathbf{SM}(\Lambda)))$ of change in log (det (SM(Λ))) caused by the change of $R(\Lambda)$ is obtained as a function of Λ as shown in Figure 6. Next, subtracting $\Delta \log (\det(\mathbf{SM}(\Lambda)))$ of Figure 6 from $\log(\det(\mathbf{SM}(\Lambda)))$ of the Figure 5 is defined by $\log (\det (\mathbf{SM}(\Lambda)))_{net}$ as follows:

$$\log (\det (\mathbf{SM}(\Lambda))) - \Delta \log (\det (\mathbf{SM}(\Lambda)))$$

$$\equiv \log (\det (\mathbf{SM}(\Lambda)))_{\text{net}}$$
(14)

Finally, $\log(\det(\mathbf{SM}(\Lambda)))_{net}$ of equation (14) gives a determinant curve where the discontinuity is removed, as shown in Figure 7. It should be noticed that the eight

Figure 7. Logarithm values of the determinant of the system matrix without discontinuity obtained by the advanced NDIF method using equations (11) and (14) for the rectangular membrane (N = 32).

eigenvalues obtained in Figure 7 naturally have the same values as those obtained in Figure 5.

Extracting mode shapes from the singular system matrix

In the section, an appropriate way of obtaining mode shapes of the membrane of interest is proposed after a deep study. The *j* th mode shape for the *j* th eigenvalue Λ_j extracted by equation (11) for the rectangular membrane (N = 32) can be obtained by utilizing equation (10) as follows. Inserting $\Lambda = \Lambda_j$ into equation (10) yields

$$\mathbf{SM}(\Lambda_i)\mathbf{v}_i = \lambda_i(\Lambda_i)\mathbf{v}_i, i = 1, 2, \dots, N$$
(15)

from which the *N* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ can be obtained. If the rank of the system matrix $\mathbf{SM}(\Lambda_j)$ is $R(\Lambda_j)$, which is given by the rank curve of the Figure 4, the *j* th mode shape can be drawn using equation (3) in which A_k 's are replaced by the elements of the $R(\Lambda_j)$ th eigenvector $\mathbf{v}_{R(\Lambda_j)}$.

On the other hand, if the *j* th mode shape belongs to low-order mode shapes, it may not be drawn by the $R(\Lambda_j)$ th eigenvector $\mathbf{v}_{R(\Lambda_j)}$. In this case, it turns out that using one of the eigenvectors with lower order than $R(\Lambda_j)$ successfully gives the *j* th mode shape, which is explained in detail in Figure 8.

Figure 8 shows the 1st–12th mode shapes of the rectangular membrane (N = 32) obtained by the advanced NDIF method using equations (3) and (15). It is confirmed that they agree well with the exact mode shapes.²² The additional information " v_{21} and R = 23"







Figure 8. Mode shapes of the rectangular membrane (N = 32) obtained by the advanced NDIF method using equations (3) and (15): (a) 1st mode (\mathbf{v}_{21} , R = 23), (b) 2nd mode (\mathbf{v}_{25} , R = 28), (c) 3rd mode (\mathbf{v}_{26} , R = 30), (d) 4th mode (\mathbf{v}_{29} , R = 32), (e) 5th mode (\mathbf{v}_{29} , R = 32), (f) 6th mode (\mathbf{v}_{31} , R = 32), (g) 7th mode (\mathbf{v}_{32} , R = 32), (h) 8th mode (\mathbf{v}_{32} , R = 32), (i) 9th mode (\mathbf{v}_{32} , R = 32), (j) 10th mode (\mathbf{v}_{32} , R = 32), (k) 11th mode (\mathbf{v}_{32} , R = 32), and (l) 12th mode (\mathbf{v}_{32} , R = 32).

given for the first mode in Figure 8 denote that the first mode shape is drawn using the 21th eigenvector v_{21} obtained from equation (15) and that the rank of $SM(\Lambda_1)$ is 23, respectively. The additional information given for each of the 2nd–12th modes is also described in the same way as the 1st mode. It is confirmed in Figure 8 that the 1st–6th mode shapes are drawn by lower-order eigenvectors than the rank of the system matrix, while the 7th–12th mode shapes are drawn by the same order eigenvectors as the rank. The reason why the orders of eigenvectors used for the first to sixth modes in Figure 8 are not equal to the rank of the system matrix is currently under investigation, and the findings will be reported in the following paper.

Verification examples

Example studies for two arbitrarily shaped membranes are conducted to verify the validity and accuracy of the advanced NDIF method established in this paper.

Arbitrarily shaped quadrilateral membrane

Figure 9 shows the geometry and boundary node locations of an arbitrarily shaped quadrilateral membrane



Figure 9. Arbitrarily shaped quadrilateral membrane discretized with 32 boundary nodes.

discretized with 32 nodes. Figure 10 shows the determinant curve obtained by the existing NDIF method. It may be observed in the curve that the lower-order eigenvalues (Λ_1 - Λ_3) are not extracted because too many nodes are used and as the result the system matrix becomes singular. The extracted eigenvalues (Λ_4 - Λ_{15}) are summarized in the second column of Table 2.

	Existing NDIF method	Advanced NDIF method	FEM (ANSYS)		
	N = 32	N = 32	2538 nodes	1337 nodes	509 nodes
Λ_1	None	4.461 (-0.04)	4.463	4.463	4.463
Λ_2	None	6.527 ([–] 0.02)	6.528	6.528	6.528
Λ_3	None	7.471 (–0.01)	7.472	7.472	7.472
Λ_4	8.878 (-0.01)	8.879 (0.00)	8.879	8.879	8.879
Λ_5	8.989 (–0.08)	8.993 ([–] 0.03)	8.996	8.996	8.997
Λ_6	10.613 (0.01)	10.613 (0.01)	10.612	10.612	10.613
Λ_7	11.009 (0.05)	11.009 (0.05)	11.004	11.004	11.004
Λ_8	II.306 (̀−0.0́3)	II.306 (̀−0.Ó́3)	11.309	11.309	11.310
Λ_9	I I.936 (–0.02)	11.936 ([–] 0.02)	11.938	11.938	11.938
Λ_{10}	13.219 (0.03)	13.219 (0.03)	13.215	13.215	13.216
Λ_{11}	I 3.370 (̀−0.2́9)	13.370 (v0.29)	13.409	13.409	13.410
Λ_{12}	13.772 (0.01)	13.772 (0.01)	13.770	13.770	13.772
Λ_{13}	I3.935 (̀−0.0́I)	I3.935 (̀−0.0́I)	13.936	13.936	13.937
Λ_{14}	14.988 (0.01) ´	14.988 (0.01) ´	14.986	14.986	14.988
Λ_{15}	I5.43I (−0.Ó3)	I5.43I (−0.0́3)	15.436	15.436	15.438

Table 2. Eigenvalues of the arbitrarily shaped quadrilateral membrane obtained by the existing NDIF method, the advanced NDIF method, and FEM (ANSYS) (parenthesized values: errors (%) relative to FEM results using 2538 nodes).



Figure 10. Logarithm values of the determinant of the system matrix obtained by the existing NDIF method for the arbitrarily shaped quadrilateral membrane (N = 32).

In order of resolve the problem that the lower-order eigenvalues are not extracted, the advanced NDIF method using equations (11) and (14) is applied to the membrane and the resulting determinant curve is shown in Figure 11. It is confirmed in the curve that the all 15 eigenvalues (Λ_1 - Λ_{15}) are successfully extracted and also the discontinuity of the curve explained in Figure 5 is eliminated. The 15 eigenvalues are summarized in the third column of Table 2 where they are compared with the eigenvalues obtained by FEM (ANSYS) using 2538 nodes. It may be said from the comparison that the advanced NDIF method gives very accurate eigenvalues within 0.03% error except for the 11th eigenvalue, which also has a very small error (0.29%). Note that it



Figure 11. Logarithm values of the determinant of the system matrix without discontinuity obtained by the advanced NDIF method using equations (11) and (14) for the arbitrarily shaped quadrilateral membrane (N = 32).

cannot be concluded that which is more accurate, the advanced NDIF method or FEM using 2538 nodes, because the membrane does not have the exact solution. Importantly, the advanced NDIF method provides results that are almost similar to those of FEM (ANSYS) using 2538 node despite using only 32 fewer nodes. In Table 1, the accuracy of the advanced NDIF method has already been verified in the rectangular membrane having the exact solution.

Figure 12 shows the 1st–15th mode shapes obtained by the advanced NDIF method using equations (3) and (15). Although the 15 modes shapes (Figure 12) use only 32 fewer nodes, it can be confirmed that they



Figure 12. Mode shapes of the arbitrarily shaped quadrilateral membrane (N = 32) obtained by the advanced NDIF method using equations (3) and (15): (a) 1st mode (\mathbf{v}_{21} , R = 23), (b) 2nd mode (\mathbf{v}_{23} , R = 28), (c) 3rd mode (\mathbf{v}_{25} , R = 29), (d) 4th mode (\mathbf{v}_{27} , R = 31), (e) 5th mode (\mathbf{v}_{29} , R = 31), (f) 6th mode (\mathbf{v}_{29} , R = 32), (g) 7th mode (\mathbf{v}_{29} , R = 32), (h) 8th mode (\mathbf{v}_{30} , R = 32), (i) 9th mode (\mathbf{v}_{30} , R = 32), (j) 10th mode (\mathbf{v}_{30} , R = 32), (k) 11th mode (\mathbf{v}_{30} , R = 32), (l) 12th mode (\mathbf{v}_{30} , R = 32), (m) 13th mode (\mathbf{v}_{30} , R = 32), (n) 14th mode (\mathbf{v}_{31} , R = 32), and (o) 15th mode (\mathbf{v}_{30} , R = 32).

Table 3. Eigenvalues of the arbitrarily shaped membrane obtained by the existing NDIF method, the advanced NDIF method, and FEM (ANSYS) (parenthesized values: errors (%) relative to FEM results using 4375 nodes).

	Existing NDIF method	Advanced NDIF method	FEM(ANSYS)		
	N = 32	N = 32	4375 nodes	1614 nodes	431 nodes
Λ_1	None	2.709 (-0.07)	2.711	2.711	2.711
Λ_2	None	4.229 ([–] 0.07)	4.232	4.232	4.232
Λ_3	None	4.358 (0.00)	4.358	4.358	4.358
Λ_4	5.570 (-0.05)	5.569 ([–] 0.07)	5.573	5.573	5.573
Λ_5	5.934 (0.00)	5.934 (0.00)	5.934	5.934	5.934
Λ_6	6.II7 (̈́—0.̈́́2)	6.II7 ([–] 0.02)	6.118	6.118	6.118
Λ_7	7.006 (-0.10)	7.006 (-0.10)	7.013	7.013	7.014
Λ_8	7.187 (̈—0.01)́	7.187 (–0.01)́	7.188	7.188	7.188
Λ_9	7.761 (–0.01)	7.761 (–0.01)	7.762	7.762	7.763
Λ_{10}	7.835 (-0.03)	7.835 (-0.03)	7.837	7.837	7.838
Λ_{11}	8.449 ([–] 0.09)	8.449 ([–] 0.09)	8.457	8.458	8.459
Λ_{12}	8.552 (–0.02)	8.552 (-0.02)	8.554	8.554	8.555
Λ_{13}	9.018 (-0.03)	9.018 (-0.03)	9.021	9.021	9.023
Λ_{14}	9.481 (̈́—0.05́́)	9.481 (–0.05)́	9.486	9.486	9.489
Λ_{15}	9.544 (–0.03)́	9.544 ([–] 0.03)	9.547	9.548	9.550



Figure 13. Mode shapes of the arbitrarily shaped quadrilateral membrane obtained by FEM (ANSYS, 2538 nodes): (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, (f) 6th mode, (g) 7th mode, (h) 8th mode, (i) 9th mode, (j) 10th mode, (k) 11th mode, (l) 12th mode, (m) 13th mode, (n) 14th mode, and (o) 15th mode.



Figure 14. Arbitrarily shaped membrane discretized with 32 boundary nodes.

exactly match those (Figure 13) obtained by FEM (ANSYS) using 2538 nodes.

Arbitrarily shaped membrane

Figure 14 shows an arbitrarily shaped membrane whose boundary is discretized with 32 nodes and consists of a



Figure 15. Logarithm values of the determinant of the system matrix obtained by the existing NDIF method for the arbitrarily shaped membrane (N = 32).

semicircle of unit radius and two equilateral edges $\sqrt{2}$ m in length. The determinant curve obtained by the existing NDIF method is shown in Figure 15 where it is may be seen that only the higher-order eigenvalues (Λ_4 - Λ_{15}) are extracted. The extracted eigenvalues (Λ_4 - Λ_{15}) are summarized in the second column of Table 3.



Figure 16. Logarithm values of the determinant of the system matrix without discontinuity obtained by the advanced NDIF method using equations (11) and (14) for the arbitrarily shaped membrane (N = 32).

The determinant curve obtained by the advanced NDIF method using equations (11) and (14) is shown in Figure 16 where it is confirmed that the all 15 eigenvalues (Λ_1 - Λ_{15}) are successfully extracted and also the discontinuity of the curve is eliminated. The 15 eigenvalues are summarized in the third column of Table 3 where they are compared with the eigenvalues calculated by FEM (ANSYS) using 4375 nodes. It may be said from the comparison that the advanced NDIF method using only 32 nodes has very small error within 0.09% with respect to FEM using 4375 nodes. For the reference, the computational times of the existing NDIF method, the advanced NDIF method, and ANSYS using 4375 nodes are approximately 81, 96, and 127s, respectively. Figures 17 and 18 show mode shapes obtained by the advanced NDIF method and FEM (ANSYS), respectively. It can be confirmed that the mode shapes by the proposed method using only 32



Figure 17. Mode shapes of the arbitrarily shaped membrane (N=32) obtained by the advanced NDIF method using equations (3) and (15): (a) 1st mode (\mathbf{v}_{19} , R = 23), (b) 2nd mode (\mathbf{v}_{23} , R = 28), (c) 3rd mode (\mathbf{v}_{23} , R = 29), (d) 4th mode (\mathbf{v}_{25} , R = 31), (e) 5th mode (\mathbf{v}_{27} , R = 32), (f) 6th mode (\mathbf{v}_{27} , R = 32), (g) 7th mode (\mathbf{v}_{27} , R = 32), (h) 8th mode (\mathbf{v}_{29} , R = 32), (i) 9th mode (\mathbf{v}_{31} , R = 32), (j) 10th mode (\mathbf{v}_{30} , R = 32), (k) 11th mode (\mathbf{v}_{31} , R = 32), (l) 12th mode (\mathbf{v}_{32} , R = 32), (m) 13th mode (\mathbf{v}_{32} , R = 32), and (o) 15th mode (\mathbf{v}_{32} , R = 32).



Figure 18. Mode shapes of the arbitrarily shaped membrane obtained by FEM (ANSYS, 4375 nodes): (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, (f) 6th mode, (g) 7th mode, (h) 8th mode, (i) 9th mode, (j) 10th mode, (k) 11th mode, (l) 12th mode, (m) 13th mode, (n) 14th mode, and (o) 15th mode.

nodes exactly matches those by FEM (ANSYS) using 4375 nodes.

Conclusion

In the paper, we have developed the advanced NDIF method that can overcome the weakness of the existing NDIF method that lower-order eigenvalues are not extracted when the number of boundary nodes increases. Furthermore, the advanced NDIF method is confirmed to maintain the high accuracy of the existing NDIF method, which has been recognized in a much more accurate method than other numerical analysis methods such as FEM and BEM. The validity and accuracy of the developed method were confirmed by two case studies. It is expected that the developed method can be applied to the eigenanalysis of arbitrarily shaped plates and acoustic cavities. In the case of membranes with other boundary conditions or with complex shapes (e.g. a membrane having a hole inside or a concave shape), further theoretical development different from the proposed method is required, and the related research is currently in progress.

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